Sequences and Series of Functions

In this lesson, we define and study the convergence of sequences and series of functions. There are many different ways to define the convergence of a sequence of functions, and different definitions lead to inequivalent types of convergence. We consider here two basic types: pointwise and uniform convergence.

Definition 0.1. *Pointwise convergence: Pointwise convergence defines the convergence of functions in terms of the conver- gence of their values at each point of their domain. Suppose that*(f_n) *is a sequence of functions* $f_n : A \to \mathbb{R}$*and* $f : A \to \mathbb{R}$. *Then* $f_n \to f$ *pointwise on* A *if* $f_n(x) \to f(x)$ *as* n *for every* $x \in A$. *We say that the sequence* (f_n) *converges pointwise if it converges pointwise to some function* f *, in which case* $f(x) = lim f_n(x)$. *Pointwise convergence is, perhaps, the most natural way to define the convergence of functions, and it is one of the most important. Nevertheless, as the following examples illustrate, it is not as well-behaved as one might initially expect.*

Example 0.2. *Suppose that* $f_n : (0, 1) \to \mathbb{R}$*is defined by* n $f_n(x) = \frac{n}{nx+1}$ *Then, since* $x \neq 0$, $lim_{n\to\infty} f_n(x) = lim_{n\to\infty} \frac{1}{x+1/n} = 1/x$. $so f_n \to f$ *pointwise* *where $f : (0,1) \to \mathbb{R}$ is given by $f(x) = 1/x$. We have $|fn(x)| < n$ for all $x \in (0,1)$, so each f_n is bounded on $(0,1)$, but their pointwise limit $f$is not. Thus, pointwise convergence does not, in general, preserve boundedness.*

Example 0.3. *Suppose that $f_n : [0,1] \to \mathbb{R}$ is defined by $f_n(x) = x^n$. If $0 \leq x < 1$, then $x^n \to 0$ as $n \to \infty$, while if$x = 1$, then $x^n \to 1$ as $n \to \infty$. So $_n \to f$ pointwise where $f(x) = 0$ if $0 \leq x < 1$, and $f(x) = 1$ if $x = 1$.*

Although each f n is continuous on [0, 1], their pointwise limit f is not (it is discon- tinuous at 1). Thus, pointwise convergence does not, in general, preserve continuity.

Example 0.4. *Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = sinnx/n$ Then $f_n \to 0$ pointwise on $\mathbb{R}$. The sequence (f_n) of derivatives $f_n(x) = cosnx$does not converge pointwise on $\mathbb{R}$; for example, $fn(\pi) = (1)^n$ does not converge as $n \to \infty$. Thus, in general, one cannot differentiate a pointwise convergent sequence. This is because the derivative of a small, rapidly oscillating function may be large.*

1 Uniform convergence

In this section, we introduce a stronger notion of convergence of functions than pointwise convergence, called uniform convergence. The difference between point- wise convergence and uniform convergence is analogous to the difference between continuity and uniform continuity.

Definition 1.1. *Suppose that* (f_n) *is a sequence of functions* $f_n : A \to \mathbb{R}$ *and* $f : A] \to \mathbb{R}$*. Then* $f_n \to f$ *uniformly on* A *if, for every* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $n > N$ *implies that* $|fn(x) - f(x)| < \epsilon$ *for all* $x \in A$*. When the domain* A *of the functions is understood, we will often say* $f_n \to f$ *uniformly instead of uniformly on* A*. The crucial point in this definition is that* N *depends only on and not on* $x \in A$*, whereas for a pointwise convergent sequence* N *may depend on both*ϵ *and* x*. A uniformly convergent sequence is always pointwise convergent (to the same limit), but the converse is not true. If for some ¿ 0 one needs to choose arbitrarily large* N *for different*$x \in A$*, meaning that there are sequences of values which converge arbitrarily slowly on* A*, then a pointwise convergent sequence of functions is not uniformly convergent.*

Definition 1.2. *The sequence* $fn(x) = x^n$ *in Example 0.3 converges pointwise on [0, 1] but not uniformly on [0, 1]. For* $0 \le x < 1$ *and* $0 < \epsilon < 1$*, we have* $|fn(x) - f(x)| = |x^n| < \epsilon$ *if and only if* $0 \le x < {}^1/n$ *. Since* ${}^1/n < 1$ *for all*$n \in \mathbb{N}$*, no* N *works for all* x *sufficiently close to 1 (although there is no difficulty at* $x = 1$*). The sequence does, however, converge uniformly on [0, b] for every* $0 \le b < 1$*; for*$0 < \epsilon < 1$*, we can take* $N = log\epsilon/logb$*.*

2 Properties of uniform convergence

In this section we prove that, unlike pointwise convergence, uniform convergence preserves boundedness and continuity. Uniform convergence does not preserve dif- ferentiability any better than pointwise convergence. Nevertheless, we give a result that allows us to differentiate a convergent sequence; the key assumption is that the derivatives converge uniformly.

- Boundedness First, we consider the uniform convergence of bounded functions.

Theorem 2.1. *Suppose that* $f_n : A \to \mathbb{R}$ *is bounded on* A *for every* $n \in \mathbb{N}$*and* $f_n \to f$*uniformly on* A*. Then* $f : A \to \mathbb{R}$ *is bounded on* A

Proof. Proof. Taking $\epsilon = 1$ in the definition of the uniform convergence, we find that there exists$N \in \mathbb{N}$ such that $|fn(x) - f(x)| < 1$ for all $x \in A$ if$n > N$. Choose some$n > N$. Then, since f n is bounded, there is a constant $M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$. It follows that $|f(x)| \leq |f(x) - fn(x)| + |fn(x)| < 1 + M_n$ for all$x \in A$, meaning that f is bounded on A (by $1 + M_n$). We do not assume here that all the functions in the sequence are bounded by the same constant. (If they were, the pointwise limit would also be bounded by that constant.) In particular, it follows that if a sequence of bounded functions converges pointwise to an unbounded function, then the convergence is not uniform. □

- continuity

One of the most important property of uniform convergence is that it preserves continuity.

Theorem 2.2. *If a sequence (f_n) of continuous functions $fn : A \to \mathbb{R}$ converges uniformly on $A\mathbb{R}$ to $f : A \to \mathbb{R}$, then f is continuous on A. Proof. Suppose that c A and ¿ 0 is given. Then, for every n N, $|f(x) - f(c)| \leq |f(x) - fn(x)| + |fn(x) - fn(c)| + |fn(c) - f(c)|$. By the uniform convergence of (f_n), we can choosen $\in \mathbb{N}$ such that $|fn(x) - f(x)| < \epsilon/3$ for all$x \in A$, and for such an n it follows that $|f(x)_f(c)| < |f_n(x) - f_n(c)| + 2\epsilon/3$ (Here we use the fact that f_n is close to f at both x and c, where x is an an arbitrary point in a neighborhood of c; this is where we use the uniform convergence in a crucial way.) Since $f_n$is continuous on A, there exists $\delta > 0$ such that $|f_n(x) - f_n(c)| < \epsilon$ if $|x - c < \delta$ and $x \in A$,which implies that $|f(x) - f(c)| < \epsilon$ if $|x - c| < \delta$ and $x \in A$. This proves that f is continuous*

This result can be interpreted as justifying an exchange in the order of limits $lim_{n\to\infty} lim_{x\to c} f_n(x) = lim_{x\to c} lim_{n\to\infty} f_n(x)$. Such exchanges of limits always require some sort of condition for their validity in this case, the uniform convergence of f n to f is sufficient, but pointwise convergence is not.

- Differentiability.

The uniform convergence of differentiable functions does not, in general, imply anything about the convergence of their derivatives or the differentiability of their limit. As noted above, this is because the values of two functions may be close together while the values of their derivatives are far apart. Thus, we have to impose strong conditions on a sequence of functions and their derivatives if we hope to prove that $f_n \to f$implies $f_n \to f$. The following example shows that the limit of the derivatives need not equal the derivative of the limit even if a sequence of differentiable functions converges uniformly and their derivatives converge pointwise.

Example 2.3. *Consider the sequence* (f_n) *of functions* $f_n : \mathbb{R} \to \mathbb{R}$*defined by* $f_n(x) = x/1 + nx^2$ *Then* $f_n \to 0$ *uniformly on* $\mathbb{R}$*. To see this, we write* $|f_n(x)| = (1/\sqrt{n})(\sqrt{n}|x|/1 + nx^2) = (1/\sqrt{n})(t/1 + t^2)$ *where* $t = \sqrt{n}|x|$*. We have* $t/1 + t^2 \leq 1/2$ *for all* $t \in \mathbb{R}$*, since* $(1 - t)^2 \geq 0$*, which implies that* $2t1 + t^2$ *. Using this inequality, we get* $|f_n(x)| \leq 1/2\sqrt{n}$ *for all* $x \in \mathbb{R}$*. Hence, given* $\epsilon > 0$*, choose* $N = 1/(4\epsilon^2)$*. Then* $|f_n(x)| < \epsilon$ *for all* $x \in \mathbb{R}$ *if* n ¿ N *, which proves that*(f_n) *converges uniformly to 0 on*$\mathbb{R}$*. (Alternatively, we could get the same result by using calculus to compute the maximum value of* $|fn| on \mathbb{R}$*.) Each* f_n *is differentiable with* $f_n(x) = (1nx^2)/(1 + nx^2)^2$ *It follows that* $f_n \to g$ *pointwise as*$n \to \infty$ *where* $g(x) = 0$ *if* $x \neq 0$ *and* $g(x) = 1$ *if* $x = 0$ *, The convergence is not uniform since g is discontinuous at 0. Thus,* $f_n \to 0$ *uni- formly, but* $f_n(0) \to 1$*, so the limit of the derivatives is not the derivative of the limit. However, we do get a useful result if we strengthen the assumptions and require that the derivatives converge uniformly, not just pointwise. The proof involves a slightly tricky application of the mean value theorem.*

Theorem 2.4. *Suppose that (f_n) is a sequence of differentiable functions $f_n : (a, b) \to \mathbb{R}$ such that $f_n \to f$ pointwise and $f_n^1 \to g$ uniformly for some $f, g : (a, b) \to \mathbb{R}$. Then f is differentiable on (a, b) and $f^1 = g$.*

Proof. □

3 Series of Function

The convergence of a series is defined in terms of the convergence of its sequence of partial sums, and any result about sequences is easily translated into a correspond- ing result about series.

Definition 3.1. *Suppose that* (f_n) *is a sequence of functions* $f_n : A \to \mathbb{R}$, *and define a sequence* (Sn) *of partial sums*$S_n : A \to \mathbb{R}$ *by* $S_n(x) = \sum_{k=1}^{n} f_k(x)$ *Then the series* $S(x) = \sum_{n=1}^{\infty} f_n(x)$ *converges pointwise to* $S : A \to \mathbb{R}$ *on* A *if* $S_n \to S$ *as* $n \to \infty$ *pointwise on* A, *and uniformly to* S *on* A *if* S_nS *uniformly on* A. *We illustrate the definition with a series whose partial sums we can compute explicitly.*

Example 3.2. *he geometric series* $\sum_{n=0}^{\infty} x^n$ *has partial sums* $S_n(x) = \sum_{k=0}^{n} x^k = 1x^n + 1/1x$ *Thus,*$S_n(x) \to 1/(1x)$ *as* $n \to \infty$ *if* $|x| < 1$ *and diverges if* $|x| \geq 1$, *meaning that* $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ *pointwise on* $(-1, 1)$. *Since 1/(1-x) is unbounded on (-1, 1), the convergence cannot be uniform. The series does, however, converges uniformly on* $[-\rho, \rho]$ *for every* $0 \leq \rho < 1$. *To prove this, we estimate for* $|x| \leq \rho$ *that* $|S_n(x) - 1/1 - x| = |x|^{(}n+1)/(1-x) = \rho^{(}n+1)/1-\rho$

Since $\rho^{(}n+1)/(1-\rho) \to 0$ *as* $n \to \infty$, *given any* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$, *depending only on* ϵ *and* ρ, *such that* $0 \leq \rho^{(}n+1)/1 - \rho < \epsilon$ *for all* $n > N$. *It follows that* $|\sum_{k=0}^{n} x^k - 1/(1-x)| \leq \epsilon$ *for all* $x[-\rho, \rho]$ *and all* $n > N$, *which proves that the series converges uniformly on* $[-\rho, \rho]$.

The Cauchy condition for the uniform convergence of sequences immediately gives a corresponding Cauchy condition for the uniform convergence of series.

Theorem 3.3. *Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$. The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} f_k(x)| < \epsilon$ for all $x \in A$ and all $n > m > N$.*

Proof. Let $S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + f_2(x) + \ + f_n(x)$. Then the sequence$(S_n)$, and therefore the series uniformly if and only if for every $\epsilon > 0$ there exists N such that $|S_n(x)S_m(x)| < \epsilon$, converges for all $x \in A$ and all $n, m > N$. Assuming $n > m$ without loss of generality, we have $S_n(x)S_m(x) = f_m + 1(x) + f_m + 2(x) + \ + f_n(x) = \sum_{k=m+1}^{n} f_k(x)$ so the result follows.

□

This condition says that the sum of any number of consecutive terms in the series gets arbitrarily small sufficiently far down the series.

4 The Weierstrass M -test

The following simple criterion for the uniform convergence of a series is very useful. The name comes from the letter traditionally used to denote the constants, or majorants, that bound the functions in the series.

Theorem 4.1. *(Weierstrass M -test).*

Let (f_n) be a sequence of functions $_n : A \to \mathbb{R}$, and suppose that for everyn $\in \mathbb{N}$ there exists a constant $Mn \geq 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$, $\sum_{n=1}^{\infty} M_n < \infty$

Then $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* A.

Proof. The result follows immediately from the observation that $\sum_{n=1}^{\infty} f_n$ is uniformly Cauchy if $\sum_{n=1}^{\infty} M_n$ is Cauchy. In detail, let $\epsilon > 0$ be given. The Cauchy condition for the convergence of a real series implies that there exists $N \in \mathbb{N}$ such that $\sum_{k=m+1}^{n} M_k(x) < \epsilon$ for all $n > m > N$. Then for all$x \in A$ and all$n > m > N$, we have $|\sum_{k=m+1}^{n} f_k(x)| < \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k(x) < \epsilon$ Thus, $\sum_{n=1}^{\infty} f_n$ satisfies the uniform Cauchy condition , so it converges uniformly. ☐

Example 4.2. *The series*$f(x) = \sum_{n=1}^{\infty}(1/2^n)cos(3^n x)$ *converges uniformly on* $\mathbb{R}$ *by the* M *-test since* $|(1/2^n)cos(3^n x)| \leq 1/2^n$, $\sum_{n=1}^{\infty} 1/2^n = 1$.

5 Courtesy

1. https://www.math.ucdavis.edu

2. lecture notes on sequence of functions and series of function.