4.6 Variation of Parameters

The **method of variation of parameters** applies to solve

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

Continuity of a, b, c and f is assumed, plus $a(x) \neq 0$. The method is important because it solves the largest class of equations. Specifically *included* are functions $f(x)$ like $\ln|x|$, $|x|$, e^{x^2}.

Homogeneous Equation. The method of variation of parameters uses facts about the homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{2}$$

The success depends upon writing the general solution of (2) as

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

where y_1, y_2 are *known functions* and c_1, c_2 are arbitrary constants. If a, b, c are constants, then the standard *recipe* for (2) finds y_1, y_2. It is known that y_1, y_2 as reported by the recipe are *independent*.

Independence. Two solutions y_1, y_2 of (2) are called **independent** if neither is a constant multiple of the other. The term **dependent** means *not independent*, in which case either $y_1(x) = cy_2(x)$ or $y_2(x) = cy_1(x)$ holds for all x, for some constant c. Independence can be tested through the **Wronskian** of y_1, y_2, defined by

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Theorem 13 (Wronskian and Independence)
The Wronskian of two solutions satisfies $a(x)W' + b(x)W = 0$, which implies **Abel's identity**

$$W(x) = W(x_0)e^{-\int_{x_0}^{x}(b(t)/a(t))dt}.$$

Two solutions of (2) are independent if and only if $W(x) \neq 0$.

The proof appears on page 183.

Theorem 14 (Variation of Parameters Formula)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let y_1, y_2 be two independent solutions of the homogeneous equation $ay'' + by' + cy = 0$ and let $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. Then the non-homogeneous differential equation

$$ay'' + by' + cy = f$$

has a particular solution

$$y_p(x) = y_1(x)\left(\int \frac{y_2(x)(-f(x))}{a(x)W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{a(x)W(x)}dx\right). \tag{4}$$

The proof is delayed to page 183.

History of Variation of Parameters. The solution y_p was discovered by varying the constants c_1, c_2 in the homogeneous solution (3), assuming they depend on x. This results in formulas $c_1(x) = \int C_1 F$, $c_2(x) = \int C_2 F$ where $F(x) = f(x)/a(x)$, $C_1(t) = \dfrac{-y_2(t)}{W(t)}$,$C_2(t) = \dfrac{y_1(t)}{W(t)}$; see the historical details on page 183. Then

$$
\begin{aligned}
y &= y_1(x)\int C_1 F + y_2(x)\int C_2 F && \text{Substitute in (3) for } c_1,\ c_2. \\
&= -y_1(x)\int y_2\frac{F}{W} + y_2(x)\int y_1\frac{F}{W} && \text{Use (??) for } C_1,\ C_2. \\
&= \int (y_2(x)y_1(t) - y_1(x)y_2(t))\frac{F(t)}{W(t)}dt && \text{Collect on } F/W. \\
&= \int \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}F(t)dt && \text{Expand } W = y_1y_2' - y_1'y_2.
\end{aligned}
$$

Any one of the last three equivalent formulas is called a **classical variation of parameters formula**. The fraction in the last integrand is called Cauchy's kernel. We prefer the first, equivalent to equation (4), for ease of use.

18 Example (Independence) Consider $y'' - y = 0$. Show the two solutions $\sinh(x)$ and $\cosh(x)$ are independent using Wronskians.

Solution: Let $W(x)$ be the Wronskian of $\sinh(x)$ and $\cosh(x)$. The calculation below shows $W(x) = -1$. By Theorem 10, the solutions are independent.

Background. The calculus *definitions* for hyperbolic functions are $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$. Their derivatives are $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$. For instance, $(\cosh x)'$ stands for $\frac{1}{2}(e^x + e^{-x})'$, which evaluates to $\frac{1}{2}(e^x - e^{-x})$, or $\sinh x$.

Wronskian detail. Let $y_1 = \sinh x$, $y_2 = \cosh x$. Then

$$
\begin{aligned}
W &= y_1(x)y_2'(x) - y_1'(x)y_2(x) && \text{Definition of Wronskian } W. \\
&= \sinh(x)\sinh(x) - \cosh(x)\cosh(x) && \text{Substitute for } y_1,\ y_1',\ y_2,\ y_2'. \\
&= \tfrac{1}{4}(e^x - e^{-x})^2 - \tfrac{1}{4}(e^x + e^{-x})^2 && \text{Apply exponential definitions.} \\
&= -1 && \text{Expand and cancel terms.}
\end{aligned}
$$

19 Example (Wronskian) Given $2y'' - xy' + 3y = 0$, verify that a solution pair y_1, y_2 has Wronskian $W(x) = W(0)e^{x^2/4}$.

Solution: Let $a(x) = 2$, $b(x) = -x$, $c(x) = 3$. The Wronskian is a solution of $W' = -(b/a)W$, hence $W' = xW/2$. The solution is $W = W(0)e^{x^2/4}$, by growth-decay theory.

20 Example (Variation of Parameters) Solve $y'' + y = \sec x$ by variation of parameters, verifying $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos(x) \ln|\cos x|$.

Solution:
Homogeneous solution y_h. The *recipe* for constant equation $y'' + y = 0$ is applied. The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ and $y_h = c_1 \cos x + c_2 \sin x$.

Wronskian. Suitable independent solutions are $y_1 = \cos x$ and $y_2 = \sin x$, taken from the *recipe*. Then $W(x) = \cos^2 x + \sin^2 x = 1$.

Calculate y_p. The variation of parameters formula (4) is applied. The integration proceeds near $x = 0$, because $\sec(x)$ is continuous near $x = 0$.

$$\begin{aligned} y_p(x) &= -y_1(x) \int y_2(x) \sec(x) dx + y_2(x) \int y_1(x) \sec x dx & \boxed{1} \\ &= -\cos x \int \tan(x) dx + \sin x \int 1 dx & \boxed{2} \\ &= x \sin x + \cos(x) \ln|\cos x| & \boxed{3} \end{aligned}$$

Details: $\boxed{1}$ Use equation (4). $\boxed{2}$ Substitute $y_1 = \cos x$, $y_2 = \sin x$. $\boxed{3}$ Integral tables applied. Integration constants set to zero.

21 Example (Two Methods) Solve $y'' - y = e^x$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution: The general solution is reported to be $y = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x/2$. Details follow.

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots ± 1. The homogeneous solution is $y_h = c_1 e^x + c_2 e^{-x}$.

Undetermined Coefficients Summary. The basic trial solution method gives initial trial solution $y = d_1 e^x$, because the RHS $= e^x$ has all derivatives given by a linear combination of the independent function e^x. The fixup rule applies because the homogeneous solution contains duplicate term $c_1 e^x$. The final trial solution is $y = d_1 x e^x$. Substitution into $y'' - y = e^x$ gives $2d_1 e^x + d_1 x e^x - d_1 x e^x = e^x$. Cancel e^x and equate coefficients of powers of x to find $d_1 = 1/2$. Then $y_p = xe^x/2$.

Variation of Parameters Summary. The homogeneous solution $y_h = c_1 e^x + c_2 e^{-x}$ found above implies $y_1 = e^x$, $y_2 = e^{-x}$ is a suitable independent pair of solutions. Their Wronskian is $W = -2$

The variation of parameters formula (11) applies:

$$y_p(x) = e^x \int \frac{-e^{-x}}{-2} e^x dx + e^{-x} \int \frac{e^x}{-2} e^x dx.$$

Integration, followed by setting all constants of integration to zero, gives $y_p(x) = xe^x/2 - e^x/4$.

Differences. The two methods give respectively $y_p = xe^x/2$ and $y_p(x) = xe^x/2 - e^x/4$. The solutions $y_p = xe^x/2$ and $y_p(x) = xe^x/2 - e^x/4$ differ by the homogeneous solution $-xe^x/4$. In both cases, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x,$$

because terms of the homogeneous solution can be absorbed into the arbitrary constants c_1, c_2.

Proof of Theorem 10: The function $W(t)$ given by Abel's identity is the unique solution of the growth-decay equation $W' = -(b(x)/a(x))W$; see page 3. It suffices then to show that W satisfies this differential equation. The details:

$W' = (y_1y_2' - y_1'y_2)'$	Definition of Wronskian.
$= y_1y_2'' + y_1'y_2' - y_1''y_2 - y_1'y_2'$	Product rule; $y_1'y_2'$ cancels.
$= y_1(-by_2' - cy_2)/a - (-by_1' - cy_1)y_2/a$	Both y_1, y_2 satisfy (2).
$= -b(y_1y_2' - y_1'y_2)/a$	Cancel common cy_1y_2/a.
$= -bW/a$	Verification completed.

The independence statement will be proved from the contrapositive: $W(x) = 0$ for all x if and only if y_1, y_2 are not independent. Technically, independence is defined relative to the common domain of the graphs of y_1, y_2 and W. Henceforth, *for all* x means for all x in the common domain.

Let y_1, y_2 be two solutions of (2), not independent. By re-labelling as necessary, $y_1(x) = cy_2(x)$ holds for all x, for some constant c. Differentiation implies $y_1'(x) = cy_2'(x)$. Then the terms in $W(x)$ cancel, giving $W(x) = 0$ for all x.

Conversely, let $W(x) = 0$ for all x. If $y_1 \equiv 0$, then $y_1(x) = cy_2(x)$ holds for $c = 0$ and y_1, y_2 are not independent. Otherwise, $y_1(x_0) \neq 0$ for some x_0. Define $c = y_2(x_0)/y_1(x_0)$. Then $W(x_0) = 0$ implies $y_2'(x_0) = cy_1'(x_0)$. Define $y = y_2 - cy_1$. By linearity, y is a solution of (2). Further, $y(x_0) = y'(x_0) = 0$. By uniqueness of initial value problems, $y \equiv 0$, that is, $y_2(x) = cy_1(x)$ for all x, showing y_1, y_2 are not independent.

Proof of Theorem 11: Let $F(t) = f(t)/a(t)$, $C_1(x) = -y_2(x)/W(x)$, $C_2(x) = y_1(x)/W(x)$. Then y_p as given in (4) can be differentiated twice using the product rule and the fundamental theorem of calculus rule $(\int g)' = g$. Because $y_1C_1 + y_2C_2 = 0$ and $y_1'C_1 + y_2'C_2 = 1$, then y_p and its derivatives are given by

$$\begin{aligned} y_p(x) &= y_1 \textstyle\int C_1F dx + y_2 \int C_2F dx, \\ y_p'(x) &= y_1' \textstyle\int C_1F dx + y_2' \int C_2F dx, \\ y_p''(x) &= y_1'' \textstyle\int C_1F dx + y_2'' \int C_2F dx + F(x). \end{aligned}$$

Let $F_1 = ay_1'' + by_1' + cy_1$, $F_2 = ay_2'' + by_2' + cy_2$. Then

$$ay_p'' + by_p' + cy_p = F_1 \int C_1F dx + F_2 \int C_2F dx + aF.$$

Because y_1, y_2 are solutions of the homogeneous differential equation, then $F_1 = F_2 = 0$. By definition, $aF = f$. Therefore,

$$ay_p'' + by_p' + cy_p = f.$$

The proof is complete.

Historical Details. The original variation ideas, attributed to Joseph Louis Lagrange (1736-1813), involve substitution of $y = c_1(x)y_1(x) + c_2(x)y_2(x)$ into (1) plus imposing an extra condition on the unknowns c_1, c_2:

$$c_1'y_1 + c_2'y_2 = 0.$$

The product rule gives $y' = c_1'y_1 + c_1y_1' + c_2'y_2 + c_2y_2'$, which then reduces to the two-termed expression $y' = c_1y_1' + c_2y_2'$. Substitution into (1) gives

$$a(c_1'y_1' + c_1y_1'' + c_2'y_2' + c_2y_2'') + b(c_1y_1' + c_2y_2') + c(c_1y_1 + c_2y_2) = f$$

which upon collection of terms becomes

$$c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) + ay_1'c_1' + ay_2'c_2' = f.$$

The first two groups of terms vanish because y_1, y_2 are solutions of the homogeneous equation, leaving just $ay_1'c_1' + ay_2'c_2' = f$. There are now two equations and two unknowns $X = c_1'$, $Y = c_2'$:

$$\begin{array}{rcrcl} ay_1'X & + & ay_2'Y & = & f, \\ y_1X & + & y_2Y & = & 0. \end{array}$$

Solving by elimination,

$$X = \frac{-y_2f}{aW}, \quad Y = \frac{y_1f}{aW}.$$

Then c_1 is the integral of X and c_2 is the integral of Y, which completes the historical account of the relations

$$c_1(x) = \int \frac{-y2(x)f(x)}{a(x)W(x)}dx, \quad c_2(x) = \int \frac{y_1(x)f(x)}{a(x)W(x)}dx.$$

Exercises 4.6

Independence. Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page 180.

1. $y'' - y = 0$

2. $y'' - 4y = 0$

3. $y'' + y = 0$

4. $y'' + 4y = 0$

5. $4y'' = 0$

6. $y'' = 0$

7. $4y'' + y' = 0$

8. $y'' + y' = 0$

9. $y'' + y' + y = 0$

10. $y'' - y' + y = 0$

11. $y'' + 8y' + 2y = 0$

12. $y'' + 16y' + 4y = 0$

13. $x^2y'' + y = 0$

14. $x^2y'' + 4y = 0$

15. $x^2y'' + 2xy' + y = 0$

16. $x^2y'' + 8xy' + 4y = 0$

Wronskian. Compute the Wronskian, up a constant multiple, without solving the differential equation.

17. $y'' + y' - xy = 0$

18. $y'' - y' + xy = 0$

19. $2y'' + y' + \sin(x)y = 0$

20. $4y'' - y' + \cos(x)y = 0$

21. $x^2y'' + xy' - y = 0$

22. $x^2y'' - 2xy' + y = 0$

Variation of Parameters. Find the general solution $y_h + y_p$ by applying a variation of parameters formula.

35. $y'' = x^2$

36. $y'' = x^3$

37. $y'' + y = \sin x$

38. $y'' + y = \cos x$

39. $y'' + y' = \ln |x|$

40. $y'' + y' = -\ln |x|$

41. $y'' + 2y' + y = e^{-x}$

42. $y'' - 2y' + y = e^{x}$

7.4 Cauchy-Euler Equation

The differential equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

is called the **Cauchy-Euler** differential equation of order n. The symbols a_i, $i = 0, \ldots, n$ are constants and $a_n \neq 0$.

The Cauchy-Euler equation is important in the theory of linear differential equations because it has direct application to **Fourier's method** in the study of partial differential equations. In particular, the second order Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

accounts for almost all such applications in applied literature.

A second argument for studying the Cauchy-Euler equation is theoretical: it is a single example of a differential equation with non-constant coefficients that has a known closed-form solution. This fact is due to a change of variables $(x, y) \longrightarrow (t, z)$ given by equations

$$x = e^t, \quad z(t) = y(x),$$

which changes the Cauchy-Euler equation into a constant-coefficient differential equation. Since the constant-coefficient equations have closed-form solutions, so also do the Cauchy-Euler equations.

Theorem 5 (Cauchy-Euler Equation)
The change of variables $x = e^t$, $z(t) = y(e^t)$ transforms the Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

into its equivalent constant-coefficient equation

$$a\frac{d}{dt}\left(\frac{d}{dt} - 1\right) z + b\frac{d}{dt} z + cz = 0.$$

The result is memorized by the general differentiation formula

$$(1) \qquad x^k y^{(k)}(x) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - k + 1\right) z(t).$$

Proof: The equivalence is obtained from the formulas

$$y(x) = z(t), \quad xy'(x) = \frac{d}{dt} z(t), \quad x^2 y''(x) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right) z(t)$$

by direct replacement of terms in $ax^2y'' + bxy' + cy = 0$. It remains to establish the general identity (1), from which the replacements arise.

The method of proof is mathematical induction. The induction step uses the chain rule of calculus, which says that for $y = y(x)$ and $x = x(t)$,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The identity (1) reduces to $y(x) = z(t)$ for $k = 0$. Assume it holds for a certain integer k; we prove it holds for $k+1$, completing the induction.

Let us invoke the induction hypothesis LHS = RHS in (1) to write

$\frac{d}{dt}\text{RHS} = \frac{d}{dt}\text{LHS}$	Reverse sides.
$= \frac{dx}{dt}\frac{d}{dx}\text{LHS}$	Apply the chain rule.
$= e^t \frac{d}{dx}\text{LHS}$	Use $x = e^t$, $dx/dt = e^t$.
$= x\frac{d}{dx}\text{LHS}$	Use $e^t = x$.
$= x\left(x^k y^{(k)}(x)\right)'$	Expand with $' = d/dx$.
$= x\left(kx^{k-1}y^{(k)}(x) + x^k y^{(k+1)}(x)\right)$	Apply the product rule.
$= k\,\text{LHS} + x^{k+1}y^{(k+1)}(x)$	Use $x^k y^{(k)}(x) = \text{LHS}$.
$= k\,\text{RHS} + x^{k+1}y^{(k+1)}(x)$	Use hypothesis LHS = RHS.

Solve the resulting equation for $x^{k+1}y^{(k+1)}$. The result completes the induction. The details, which prove that (1) holds with k replaced by $k+1$:

$$\begin{aligned} x^{k+1}y^{(k+1)} &= \frac{d}{dt}\text{RHS} - k\,\text{RHS} \\ &= \left(\frac{d}{dt} - k\right)\text{RHS} \\ &= \left(\frac{d}{dt} - k\right)\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k + 1\right)z(t) \\ &= \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k\right)z(t) \end{aligned}$$

1 Example (How to Solve a Cauchy-Euler Equation) Show the solution details for the equation

$$2x^2y'' + 4xy' + 3y = 0,$$

verifying general solution

$$y(x) = c_1x^{-1/2}\cos\left(\frac{\sqrt{5}}{2}\ln|x|\right) + c_2e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Solution: The characteristic equation $2r(r-1) + 4r + 3 = 0$ can be obtained as follows:

$2x^2y'' + 4xy' + 3y = 0$	Given differential equation.

$2x^2r(r-1)x^{r-2} + 4xrx^{r-1} + 3x^r = 0$	Use **Euler's substitution** $y = x^r$.
$2r(r-1) + 4r + 3 = 0$	Cancel x^r. **Characteristic equation** found.
$2r^2 + 2r + 3 = 0$	Standard quadratic equation.
$r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$	Quadratic formula complex roots.

Cauchy-Euler Substitution. The second step is to use $y(x) = z(t)$ and $x = e^t$ to transform the differential equation. By Theorem 5,

$$2(d/dt)^2 z + 2(d/dt)z + 3z = 0,$$

a constant-coefficient equation. Because the roots of the characteristic equation $2r^2 + 2r + 3 = 0$ are $r = -1/2 \pm \sqrt{5}i/2$, then the Euler solution atoms are

$$e^{-t/2}\cos\left(\frac{\sqrt{5}}{2}t\right), \quad e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}t\right).$$

Back-substitute $x = e^t$ and $t = \ln|x|$ in this equation to obtain two independent solutions of $2x^2y'' + 4xy' + 3y = 0$:

$$x^{-1/2}\cos\left(\frac{\sqrt{5}}{2}\ln|x|\right), \quad e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Substitution Details. Because $x = e^t$, the factor $e^{-t/2}$ is written as $(e^t)^{-1/2} = x^{-1/2}$. Because $t = \ln|x|$, the trigonometric factors are back-substituted like this: $\cos\left(\frac{\sqrt{5}}{2}t\right) = \cos\left(\frac{\sqrt{5}}{2}\ln|x|\right)$.

General Solution. The final answer is the set of all linear combinations of the two preceding independent solutions.

Exercises 7.4

Cauchy-Euler Equation. Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page 452.

1. $x^2y'' + y = 0$

2. $x^2y'' + 4y = 0$

3. $x^2y'' + 2xy' + y = 0$

4. $x^2y'' + 8xy' + 4y = 0$

Variation of Parameters. Find a solution y_p using a variation of parameters formula.

5. $x^2y'' = e^x$

6. $x^3y'' = e^x$

7. $y'' + 9y = \sec 3x$

8. $y'' + 9y = \csc 3x$

Courtesy (Contents are sourced from) **: ---**

1. Variation of Parameters
http://www.math.utah.edu/~gustafso/2250variation-of-parameters.pdf

2. Cauchy-Euler Equation
https://www.math.utah.edu/~gustafso/s2014/3150/slides/cauchy-euler-de.pdf